Integrable Systems vs Deterministic Chaos

Based on work with F. Popov and J. Sonnenschein [2211.14150]

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ITMP

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Liouville theorem

Analytic solutions (e.g. quadratures)

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Regular trajectories (e.g. periodicity)

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Phase space is foliated in lower-dimensional spaces (dynamics takes place on n-tori)

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Regular trajectories (e.g. periodicity)

Liouville-Arnol'd theorem

Phase space is foliated in lower-dimensional spaces (dynamics takes place on n-tori)

Structure is "not stiff": small perturbations preserve the foliation almost everywhere

Kolmogorov-Arnol'd-Moser theorem

However large perturbations might be radically different

$$\frac{d}{dt}E = 0\,, \quad E = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta)$$

$$\frac{d}{dt}E = 0\,, \quad E = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta) \quad \Longrightarrow \quad \int_{\theta_0}^{\theta(t)} \frac{d\vartheta}{\sqrt{\cos\vartheta + \frac{E}{mgl} - 1}} = \int_0^t dt\sqrt{2gl^3} = \sqrt{2gl^3}\,t$$

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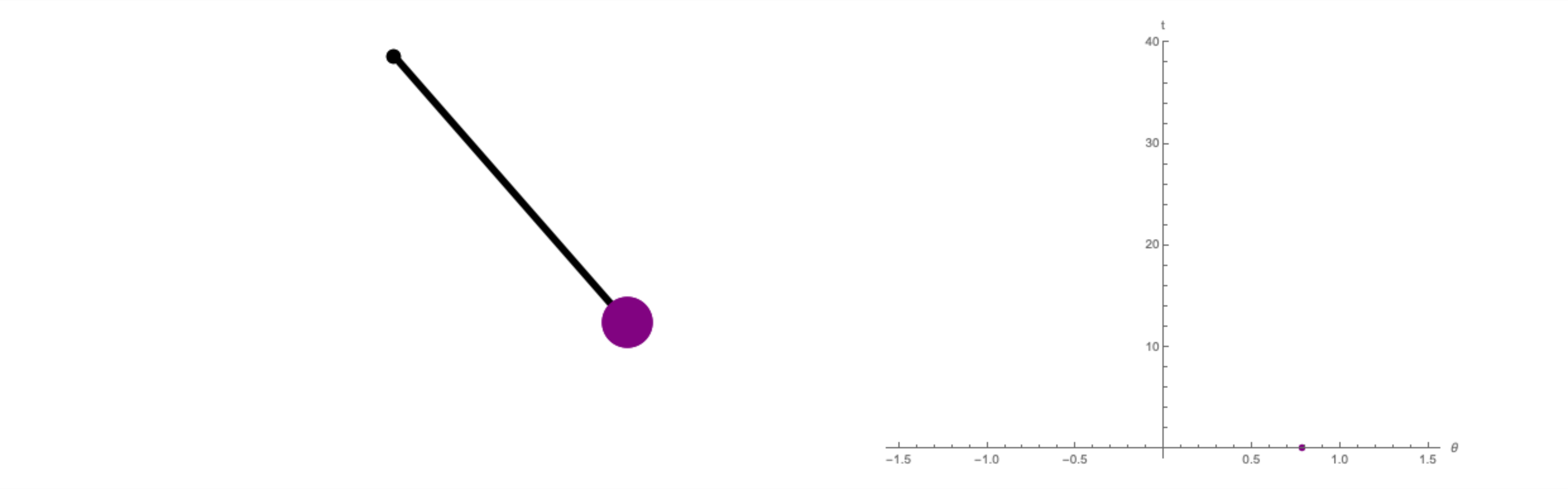
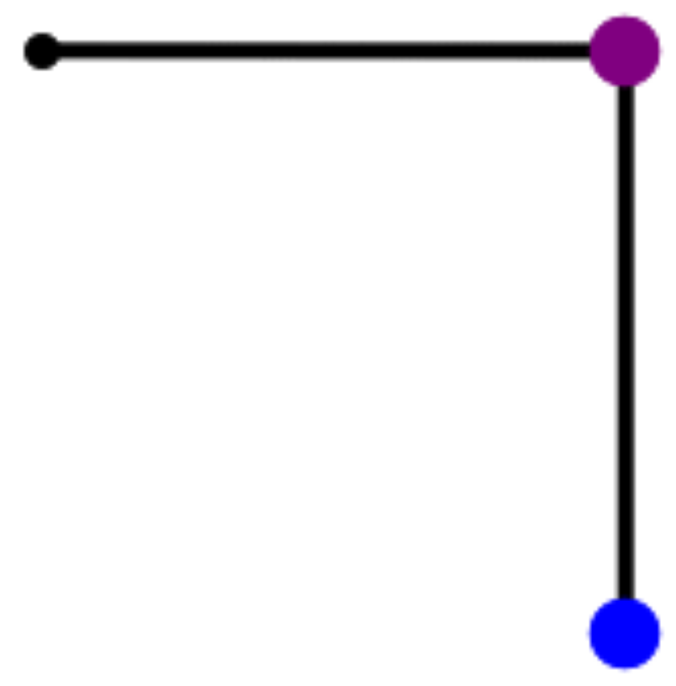
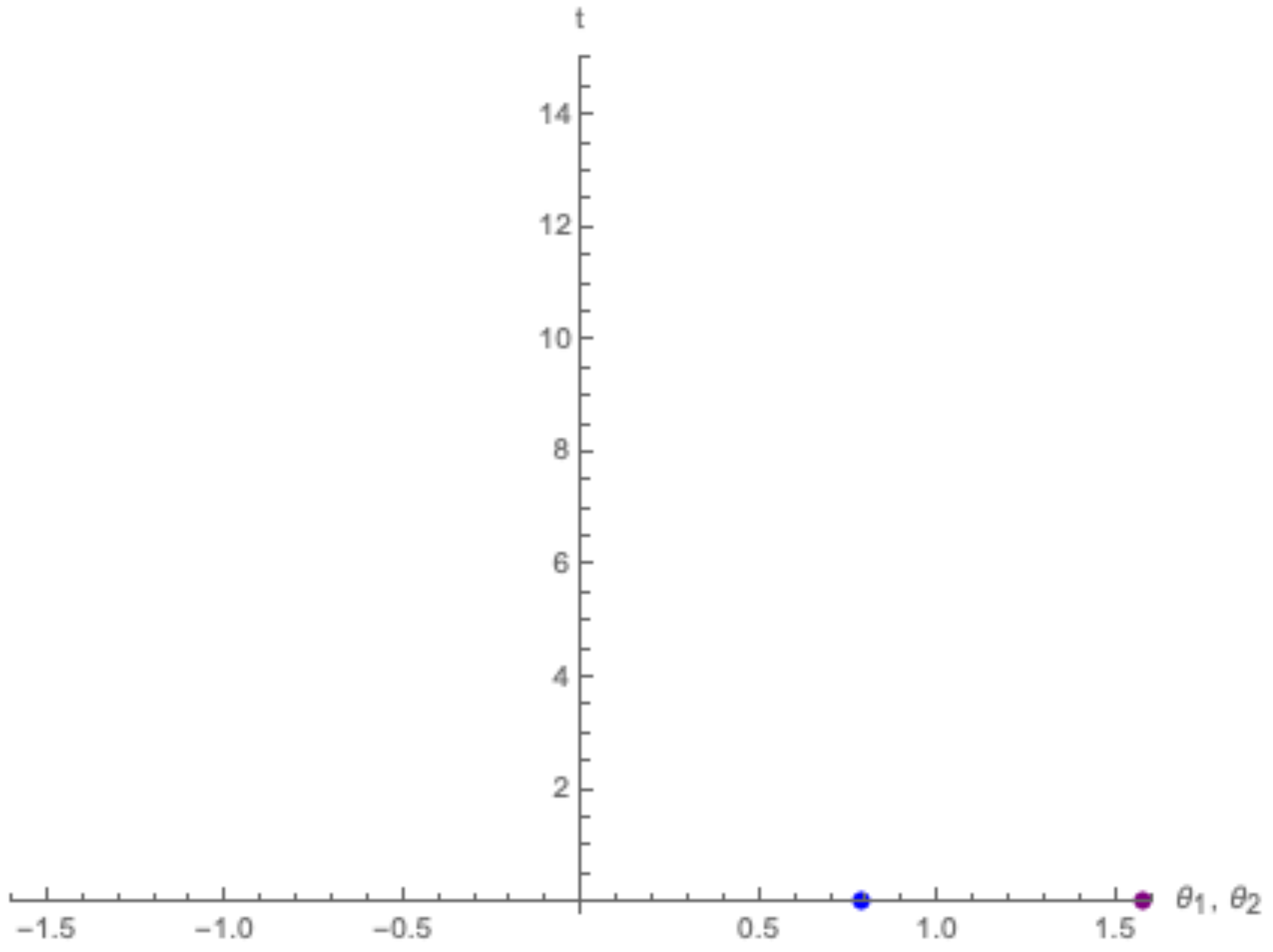
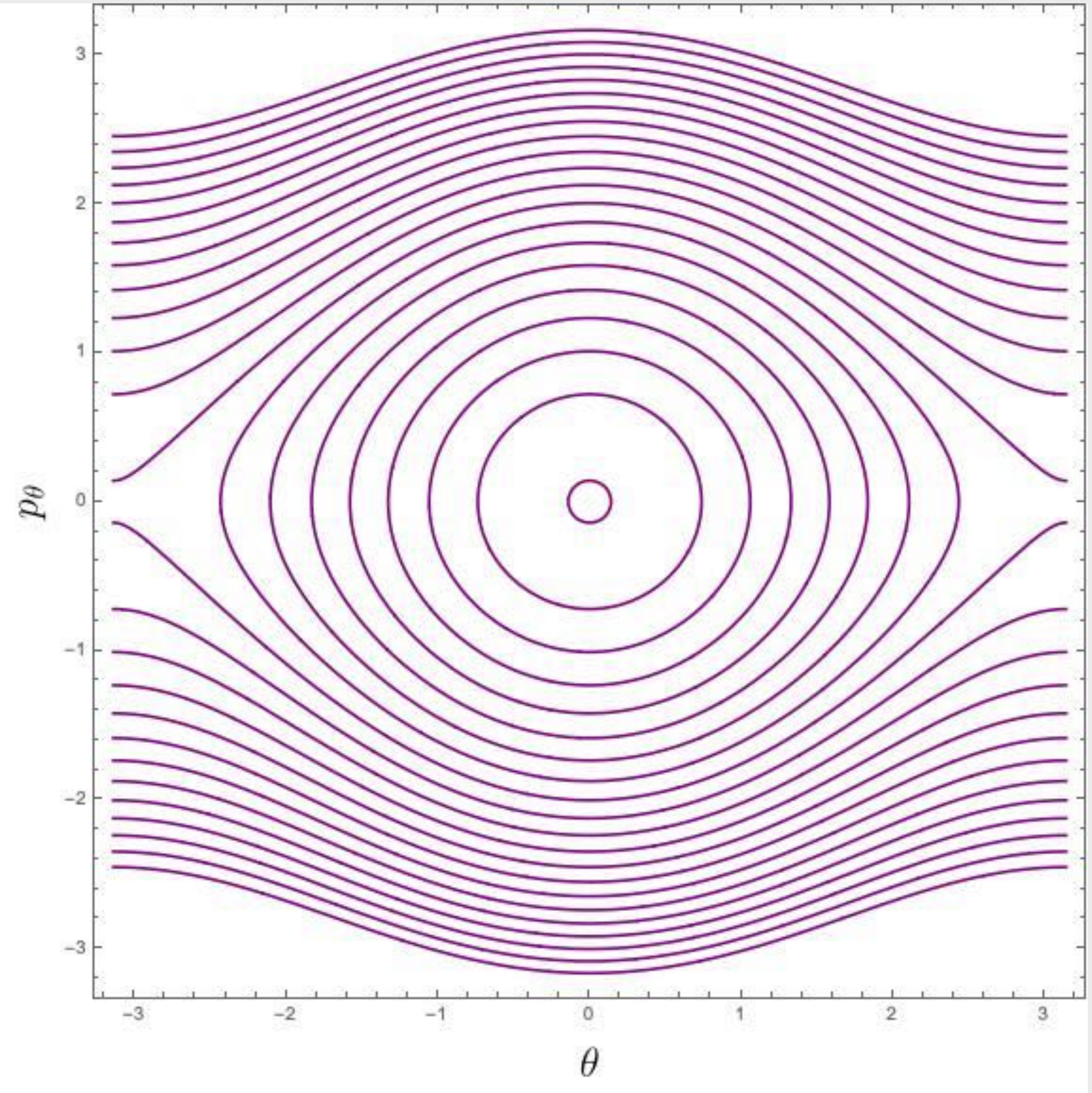
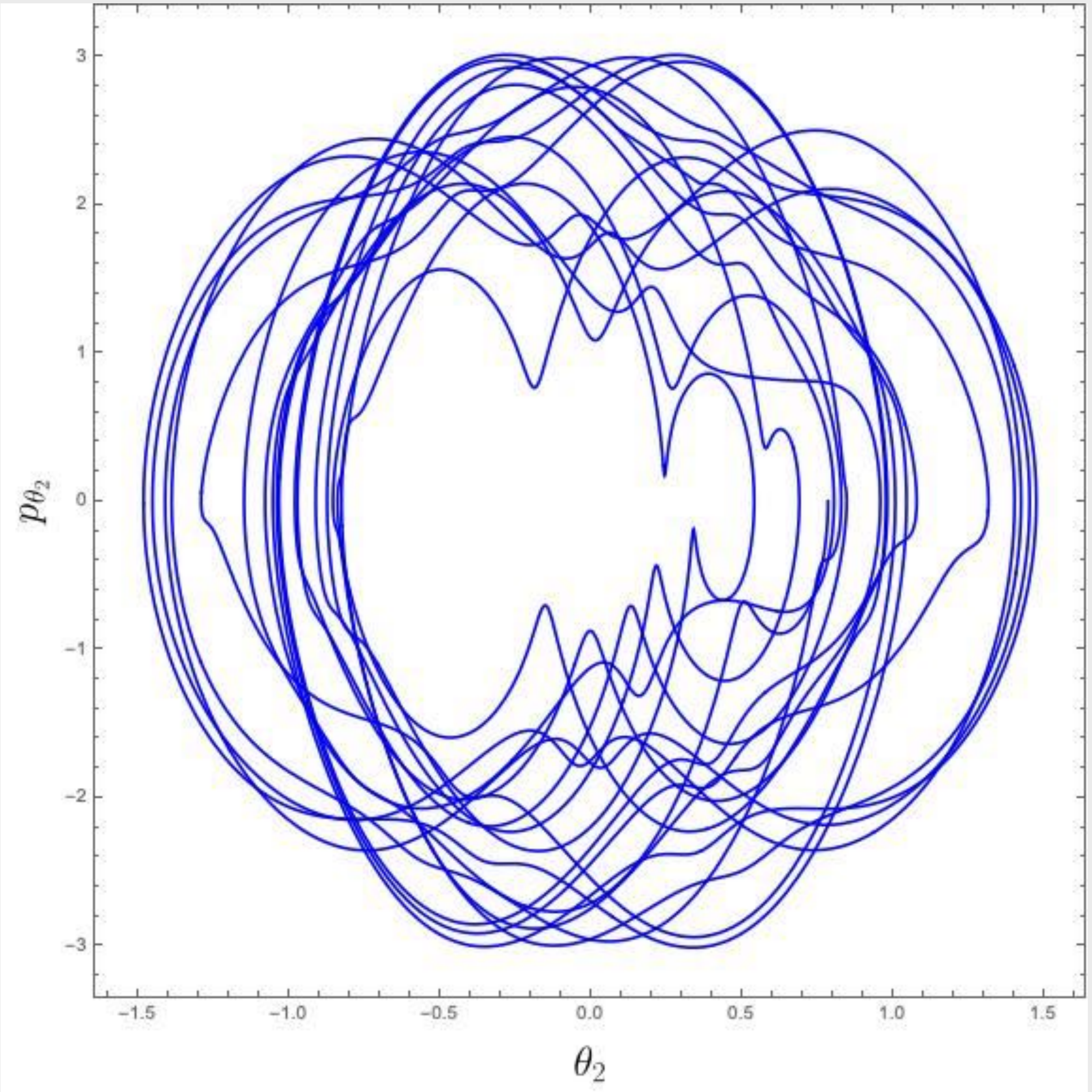


$$\frac{d}{dt}E = 0\,, \quad E = \frac{1}{1+\frac{m_2}{m_1}\sinh^2\left(\theta_1-\theta_2\right)}\frac{p_{\theta_1}}{2m_1 l_1^2} + \frac{1+\frac{m_2}{m_1}}{1+\frac{m_2}{m_1}\sinh^2\left(\theta_1-\theta_2\right)}\frac{p_{\theta_2}}{2m_2 l_2^2} - \frac{\cos\left(\theta_1-\theta_2\right)}{1+\frac{m_2}{m_1}\sinh^2\left(\theta_1-\theta_2\right)}\frac{p_{\theta_1}p_{\theta_2}}{l_1 l_2} - m_1 g l_1\left(1+\frac{m_2}{m_1}\right)\cos\theta_2 - m_2 g l_2 \cos\theta_2$$






p_θ
θ
-3
-2
-1
0
1
2
3


p_{θ_2}
θ_2
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
-3
-2
-1
0
1
2
3

Deterministic Chaos

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"When the present determines the future, but the approximate present does not approximately determine the future."

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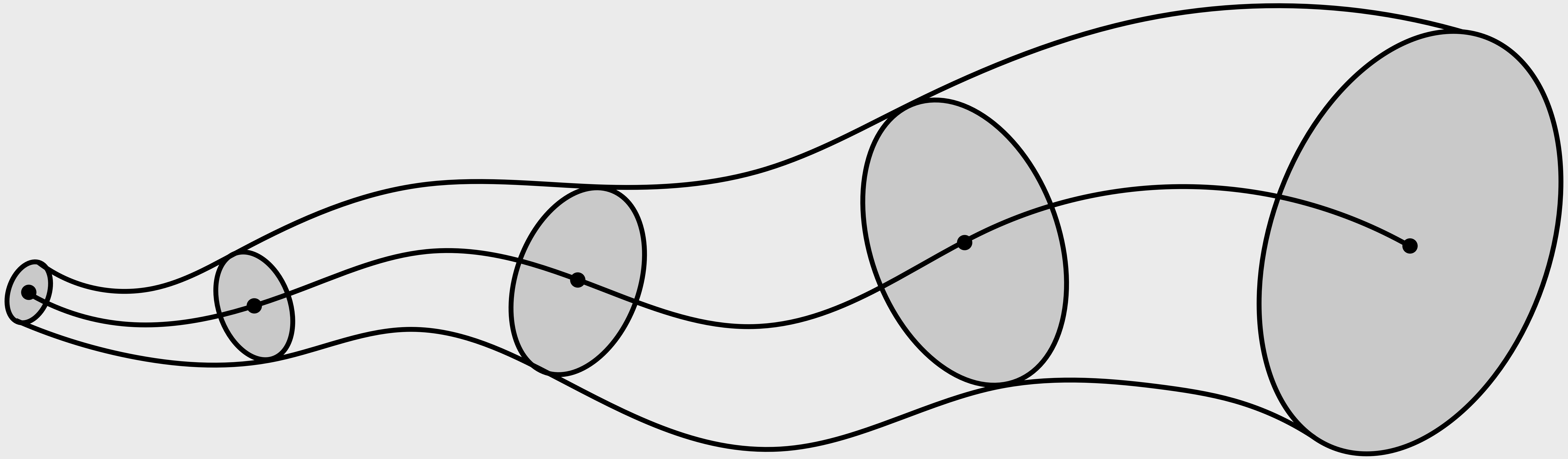
Sensitivity to initial conditions: Trajectories starting near each other, separate exponentially with time



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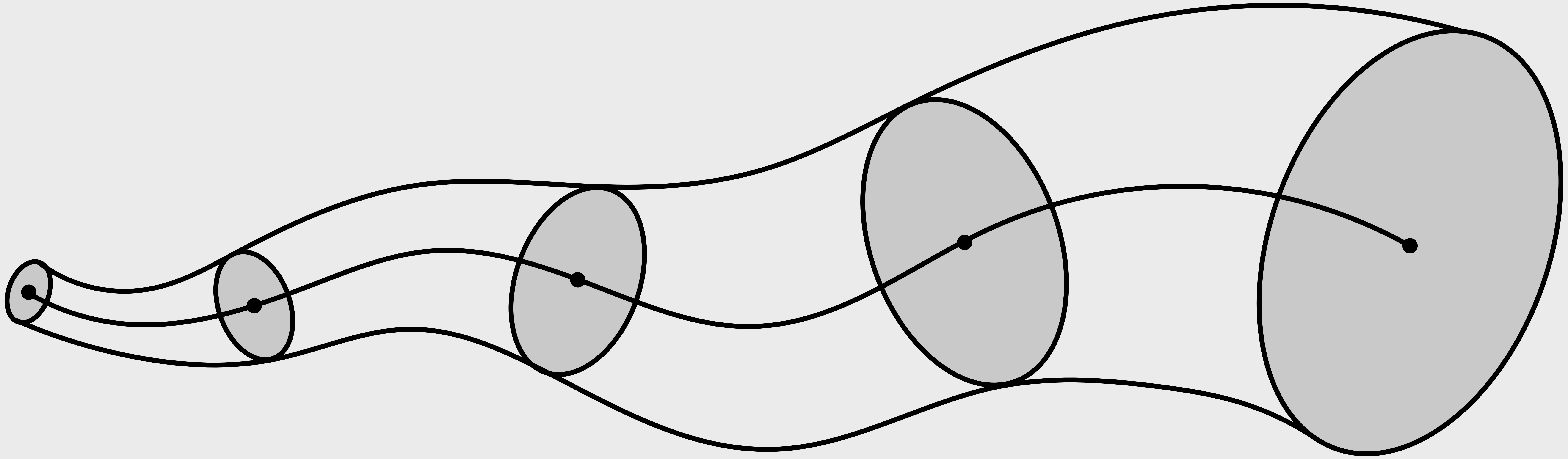
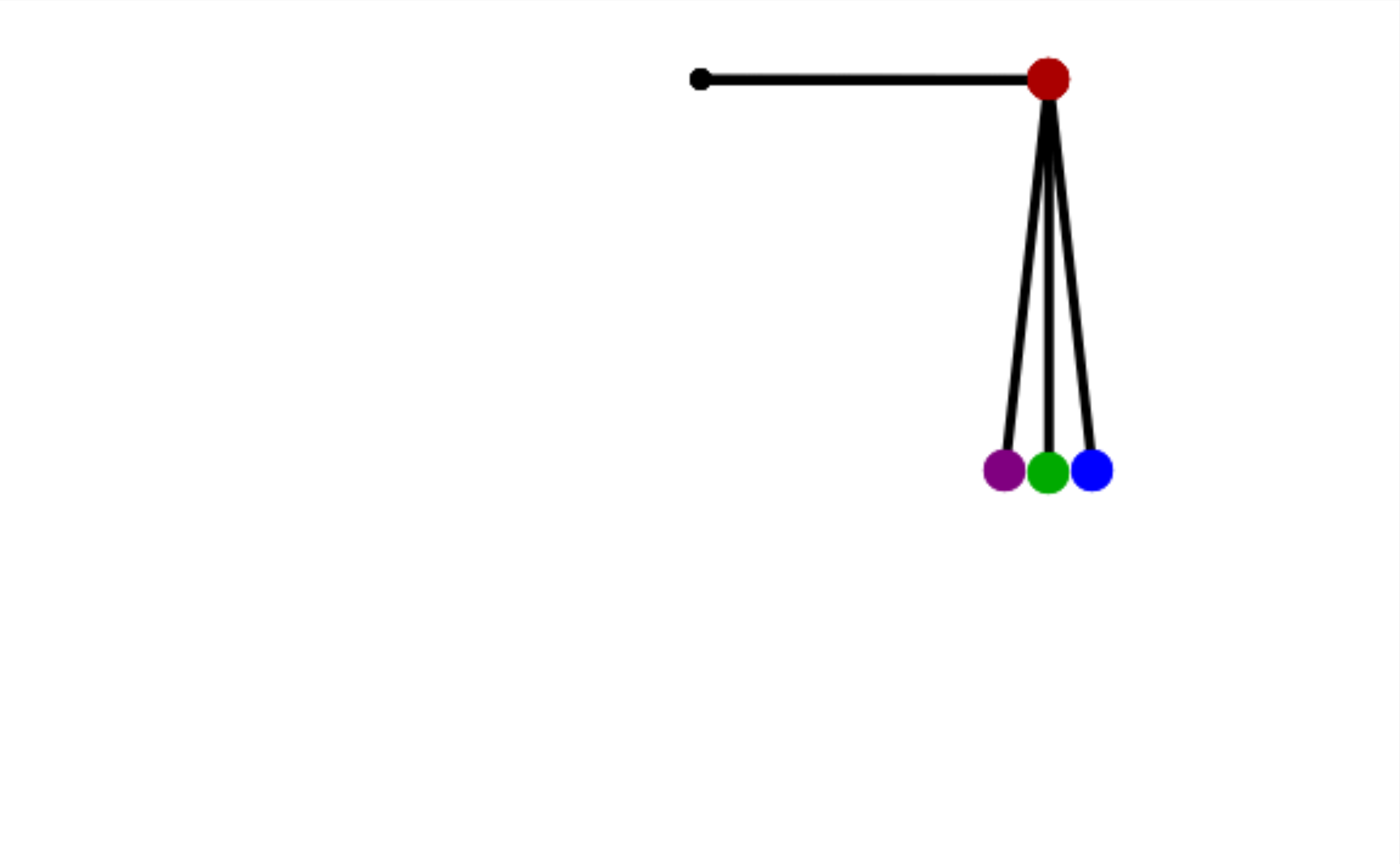
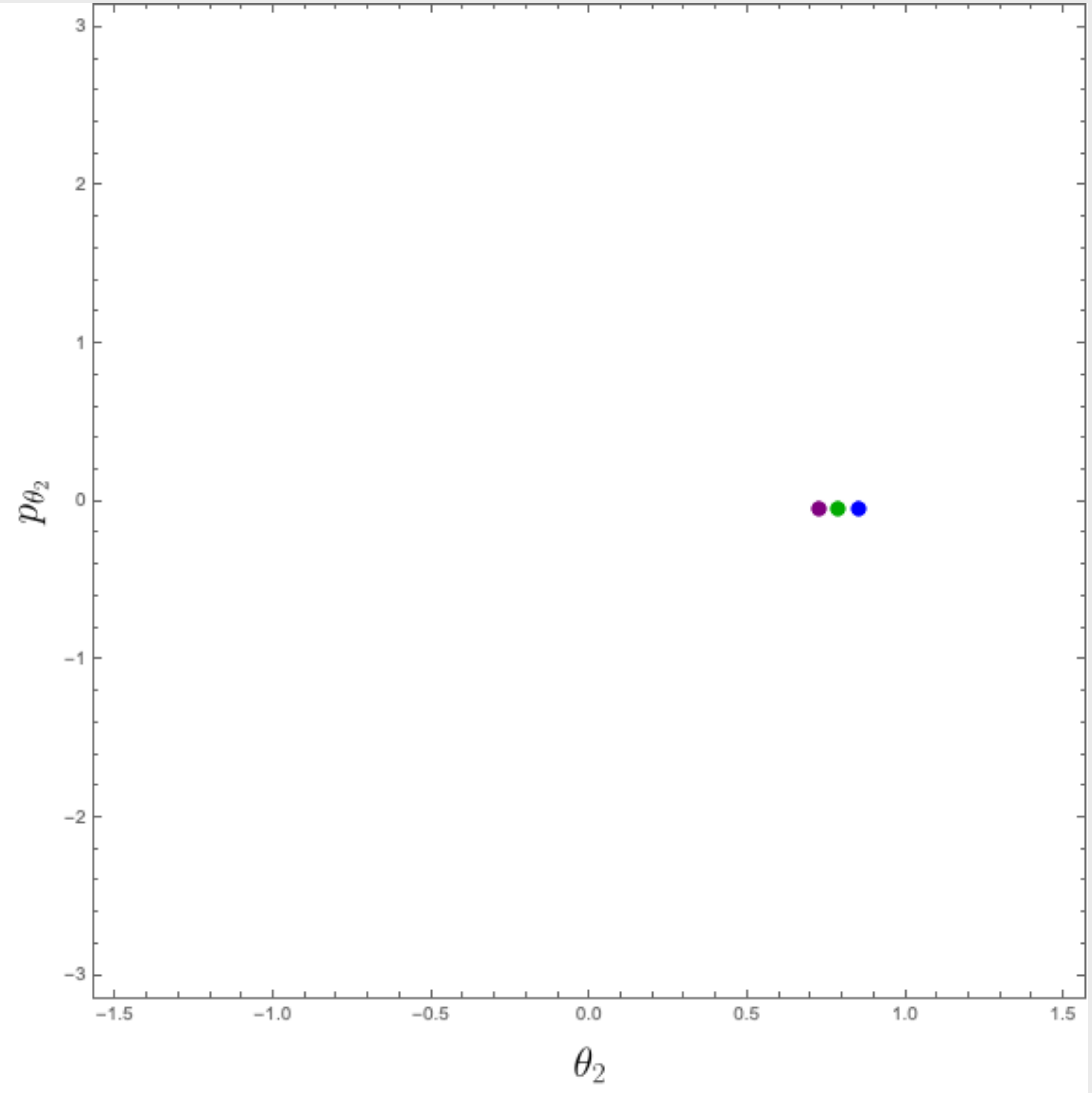


$$\left|\delta\boldsymbol{x}(t)\right| \approx e^{\lambda_{\mathrm{L}} t}\left|\delta\boldsymbol{x}(0)\right|$$

up to characteristic size of the system

Aleksandr Mikhailovich Lyapunov





3
2
1
0
−1
−2
−3
p_{θ_2}
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
θ_2

Apparent tension between integrability and chaos

Integrability $\longleftrightarrow$ regularity, predictability, periodicity

Chaos $\longleftrightarrow$ irregularity, non-predictability, absence of recurrence

Integrable systems do not thermalize on Gibbs Ensemble

[Srednicki '94 for finite D systems]

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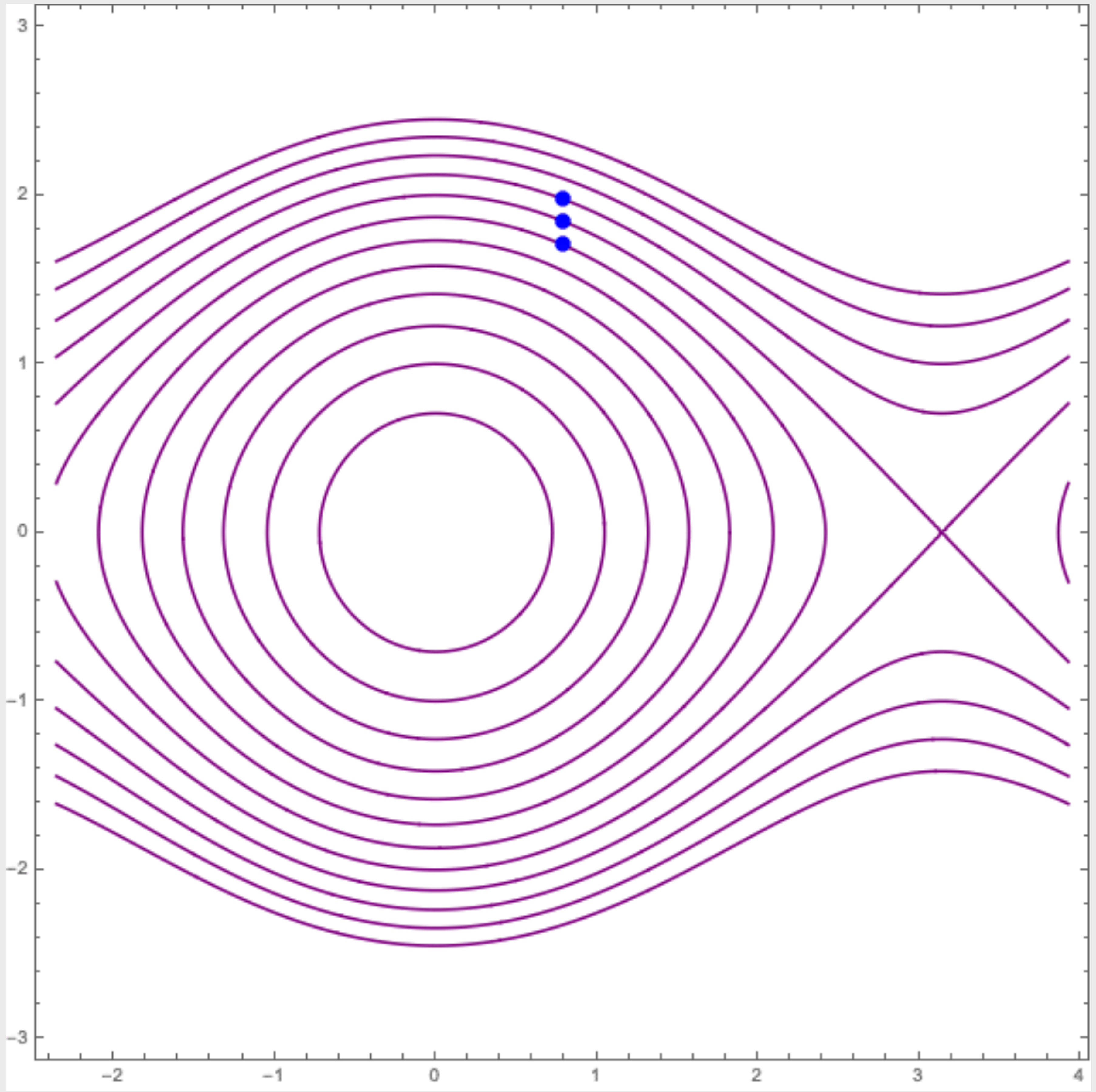
Careful definitions are important!

Integrable systems can present diverging trajectories (saddles)



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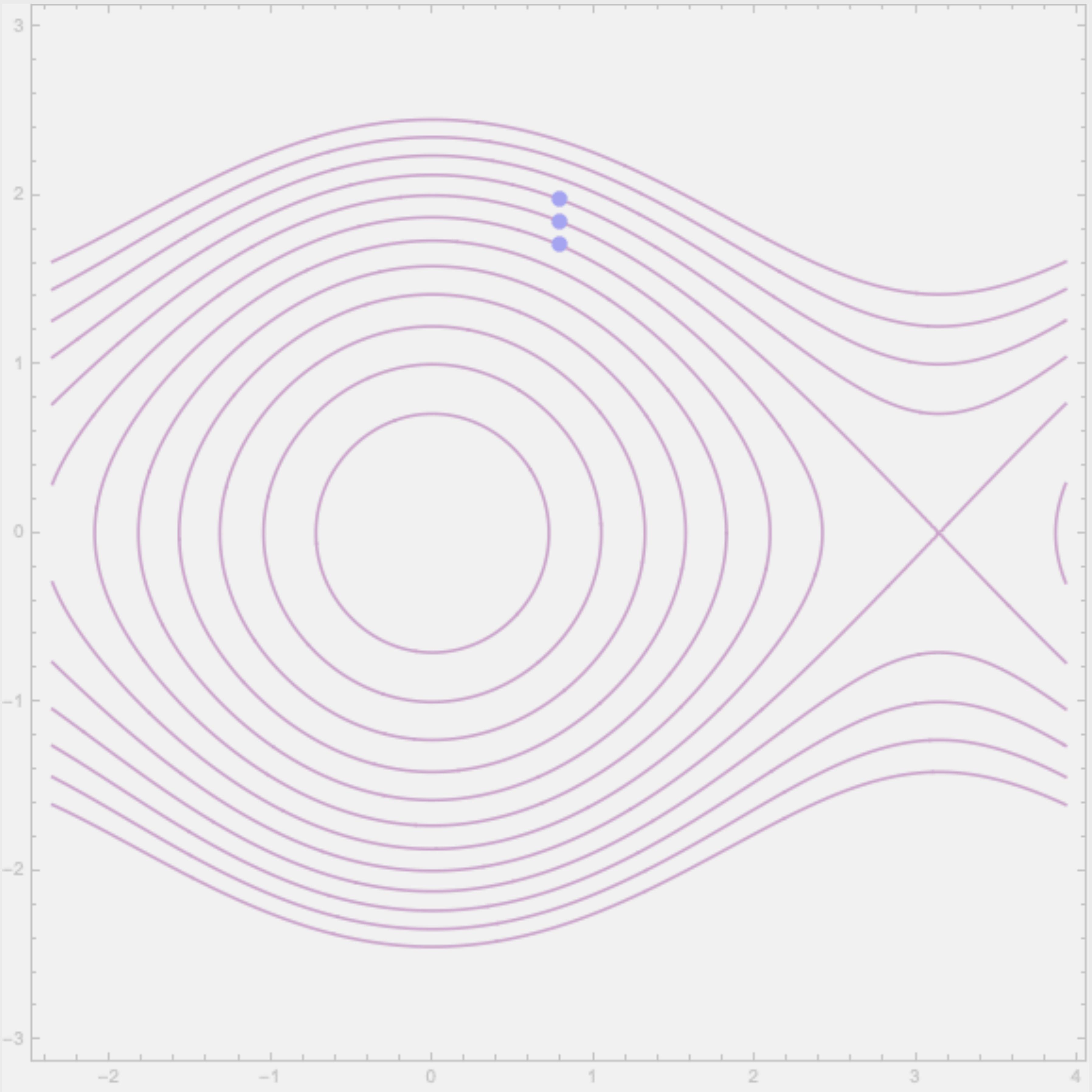
[Srednicki '94 for finite D systems]

Careful definitions are important!

Integrable systems can present diverging trajectories (saddles)

How do we tell them apart?



Integrability

there exists a method that allows one to solve the theory – i.e. find all physical observables – with a finite number of quadradures [1] or algebraic manipulations.

[Babelon, Bernard, Talon, '03]

Deterministic Chaos

for any initial state, one can find a small deformation that drives a system away under time evolution at least in a weak sense: the deviation grows and is unbounded

[Guckenheimer, Holmes, '13]

Chaotic map

a map f between two metric spaces $\left(d_1, M_1\right)$ and $\left(d_2, M_2\right)$ is chaotic if there exist nearby points in M_1 that can be sent to distant points in M_2.

$$\forall \epsilon > 0\,, \forall L > 0\,, \quad \exists x, y \in M_1 \;:\; d_1(x,y) < \epsilon\,,\, d_2\left(f(x), f(y)\right) > L$$

Finite dimensional integrable system $\iff \exists \left\{ I_j \right\}_{j=1}^{n}$ s.t. $\left\{ I_i, I_j \right\}_P = 0$

Canonical AA map $\left\{ p_j, q_j \right\}_{j=1}^{n} \longmapsto \left\{ I_j, \varphi_j \right\}_{j=1}^{n}$ s.t. $\dot{I}_j = 0\,,\ \dot{\varphi}_j = \nu_j$ and $\varphi \sim \varphi + 2\pi$

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Suppose phase space is a differentiable manifold $\Longrightarrow$ AA map is finite-dimensional & smooth

$\Longrightarrow$ AA map cannot yield arbitrarily large distances from small ones

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Finite-D ISs with phase space a differentiable manifold cannot exhibit deterministic chaos

Relax the differentiable manifold hypothesis

Then a finite dimensional integrable system
can display diverging trajectories

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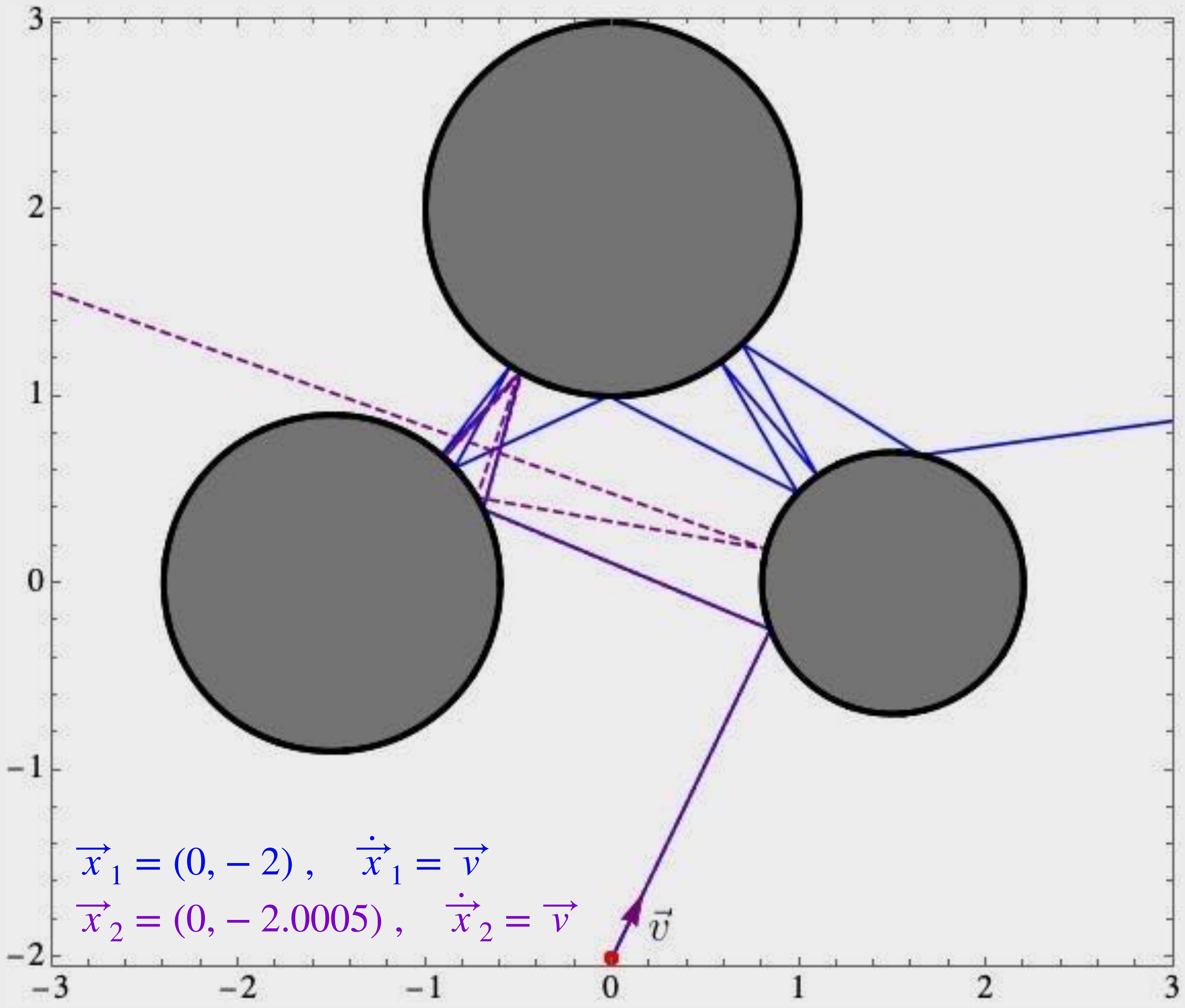
Then a finite dimensional integrable system can display diverging trajectories

Typical example: the pinball problem

Trajectories are piecewise linear: determined by finite algebraic manipulations

Phase space is <u>not smooth</u>



Baker's map

$$B \;:\; [0,1)^2 \longrightarrow [0,1)^2 \;,$$

$$B(x,y) = \begin{cases} \left(2x, \frac{y}{2}\right), & x \in \left[0, \frac{1}{2}\right) \\ \left(2 - 2x, 1 - \frac{y}{2}\right), & x \in \left[\frac{1}{2}, 1\right) \end{cases}$$

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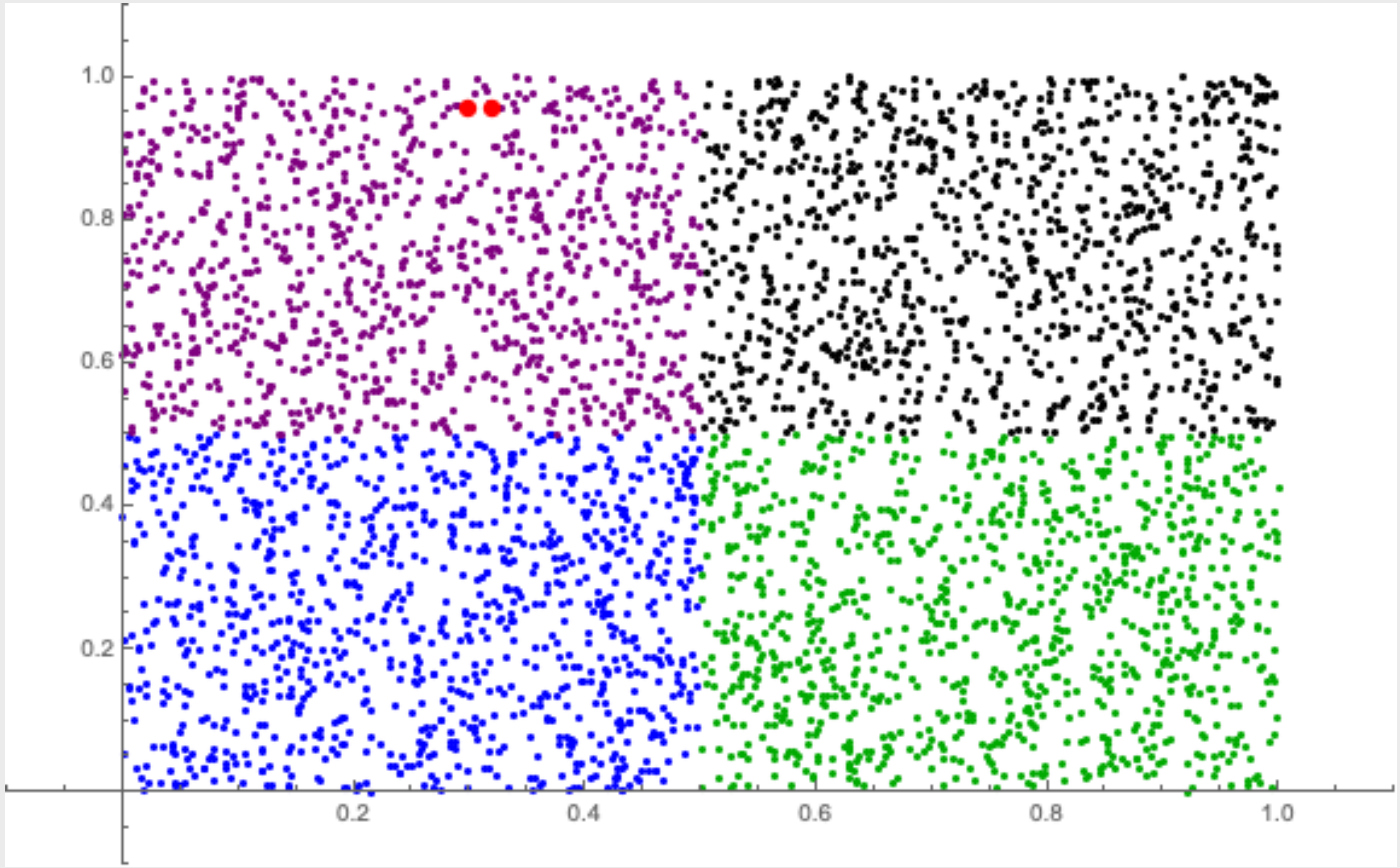
The evolution of this map is chaotic.



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This system is integrable!

$$I \;:\; \left(x, y\right) \longmapsto \left\{\sigma_i\right\}_{i\in\mathbb{Z}}$$

$$x = \sum_{i=0}^{\infty} \frac{\sigma_{-i}}{2^{i+1}} \,, \quad y = \sum_{i=0}^{\infty} \frac{\sigma_{i+1}}{2^{i+1}}$$

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$$\Downarrow$$

$$I \circ B \circ I^{-1} \ : \ \{\sigma_i\} \longmapsto \{\tilde{\sigma}_i = \sigma_{i+1}\}$$

Any totally symmetric function of σ_i is conserved

Baker's map

$I \circ B \circ I^{-1}$ maps small intervals to small intervals

The dynamics is non-chaotic in binary space

The map I injects chaos in the system

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$$I\left(\frac{1}{2},\frac{81}{256}\right) = \left\{\cdots \overset{0}{1}, \overset{1}{0}, \overset{2}{1},0,1,0,0,0,1,0,0,0,0\cdots\right\}$$

$$I\left(\frac{1}{2},\frac{1}{\pi}\right) = \left\{\cdots \overset{0}{1}, \overset{1}{0}, \overset{2}{1},0,1,0,0,0,1,0,1,1,1\cdots\right\}$$

$$\left|\frac{1}{\pi} - \frac{81}{256}\right| \approx 2\cdot 10^{-3}$$

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$$\partial_\tau u(x,\tau) + 6u(x,\tau)\partial_x u(x,\tau) + \partial_x^3 u(x,\tau) = 0$$

Solvable by *inverse scattering transform*

$$\partial_\tau u(x,\tau) + 6u(x,\tau)\partial_x u(x,\tau) + \partial_x^3 u(x,\tau) = 0$$

Solvable by *inverse scattering transform*

Start with an auxiliary scattering problem

$$\left(-\partial_x^2 + u(x)\right)\psi_k(x) = k^2\psi_k(x)$$

$$\psi_k(x) \sim \begin{cases} e^{-ikx} & x \to -\infty \\ \frac{1}{t_k}e^{-ikx} + \frac{r_k}{t_k}e^{ikx} & x \to +\infty \end{cases}$$

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If u evolves with KdV the data evolves as

$$r_k(\tau) = r_k(0)e^{8ik^3\tau}\,, \quad t_k(\tau) = t_k(0)$$

(need to consider bound states as well

$$k = i\kappa_n\,, \quad b_n = r_{i\kappa_n}$$

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Reconstruct the potential $u(x,\tau)$

(Gel′fand-Levitan-Marchenko integral equation)

$$\partial_\tau u(x,\tau) + 6u(x,\tau)\partial_x u(x,\tau) + \partial_x^3 u(x,\tau) = 0$$

Solvable by *inverse scattering transform*

Acts as a "non-linear Fourier transform"

Initial configuration $u(x, \tau = 0)$ → (Direct scattering transform) → Scattering data $\{r_k, t_k, b_n\}$

Initial configuration ↓ (Evolution in field space) ↓ Evolved configuration

Scattering data ↓ (Evolution in scattering space) ↓ Evolved scattering data

Evolved scattering data $\{r_k(\tau), t_k(\tau), b_n(\tau)\}$ → (Inverse scattering transform) → Evolved configuration $u(x, \tau)$

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Evolution in scattering space

Evolved configuration $u(x,\tau)$ ← Inverse scattering transform ← Evolved scattering data $\{r_k(\tau), t_k(\tau), b_n(\tau)\}$

These maps are chaotic!

$$\partial_\tau u(x,\tau) + 6u(x,\tau)\partial_x u(x,\tau) + \partial_x^3 u(x,\tau) = 0$$

Solvable by *inverse scattering transform*

Acts as a "non-linear Fourier transform"

Well known in geophysics (!)

[Carrion, '85 | Dorren, Muyzert, Snieder, '94]

Initial configuration $u(x,\tau=0)$ → Direct scattering transform → Scattering data $\{r_k, t_k, b_n\}$

Evolution in field space ↓ | Evolution in scattering space ↓

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These maps are chaotic!

Direct scattering transform is chaotic

Any small "well" in $u(x)$ corresponds to a bound state $\Longrightarrow$ drastic modification of $\{t_k, r_k, \kappa_n, b_n\}$
[Landau, Lifshitz #3]

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Scattering in 1D (and 2D) is "strongly-coupled"

Small potential $u(x) \ll 1$

$$\psi_k(x) = e^{-ikx} + \delta\psi_k(x)$$

$$\left(\partial_x^2 + k^2\right) \delta\psi_k(x) = u(x)e^{-ikx}$$

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$\Longrightarrow$

$$\delta\psi_k(x) = \int \frac{dy}{4\pi i} \frac{e^{ik|x-y|}}{k} u(y) e^{-iky}$$

Arbitrarily large corrections from the IR $k \sim 0$

Gain a concrete feeling of the instability computing conserved charges

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Definition via recursive relation

$$Q_n = \int_{\mathbb{R}} dx\, w_{n-1}(x)$$

$$\begin{cases} w_0(x) = u(x) \\ w_n(x) = \partial_x w_{n-1}(x) + \sum_{k=0}^{n-2} w_k(x) w_{n-2-k}(x) \end{cases}$$

$w_{2n-1}(x)$ are total derivatives $\Rightarrow$ $Q_{2n} = 0$

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$w_{2n-1}(x)$ are total derivatives $\Rightarrow$ $Q_{2n} = 0$

	$u(x) = 0$	$u(x) = \frac{e^{-x^2}}{10}$
Q_1	0	1.772×10^{-1}
Q_5	0	-1.049×10^{-2}
Q_9	0	-1.527×10^{-1}
Q_{13}	0	-9.066
Q_{17}	0	-1.186×10^{3}

Similar situation as with the Baker's map

The dynamics is non-chaotic in scattering space

The inverse/direct scattering maps inject chaos

$$\begin{cases} w_0(x) = u(x) \\ w_n(x) = \partial_x w_{n-1}(x) + \sum_{k=0}^{n-2} w_k(x) w_{n-2-k}(x) \end{cases}$$

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- Integrable systems can show behaviour typically associated with chaos
  - The map from initial conditions to conserved charges is responsible

- Integrability vs chaos in quantum systems

- Explore more refined measures of chaos (e.g. statistics of trajectories)

- Systems with non-differentiable phase space from high-energy theory (Zig-zag, TTbar, …)

Thank you

$$\partial_t\phi(x,t) + 3\phi(x,t)^2\partial_x\phi(x,t) + \partial_x^3\phi(x,t) = 0$$

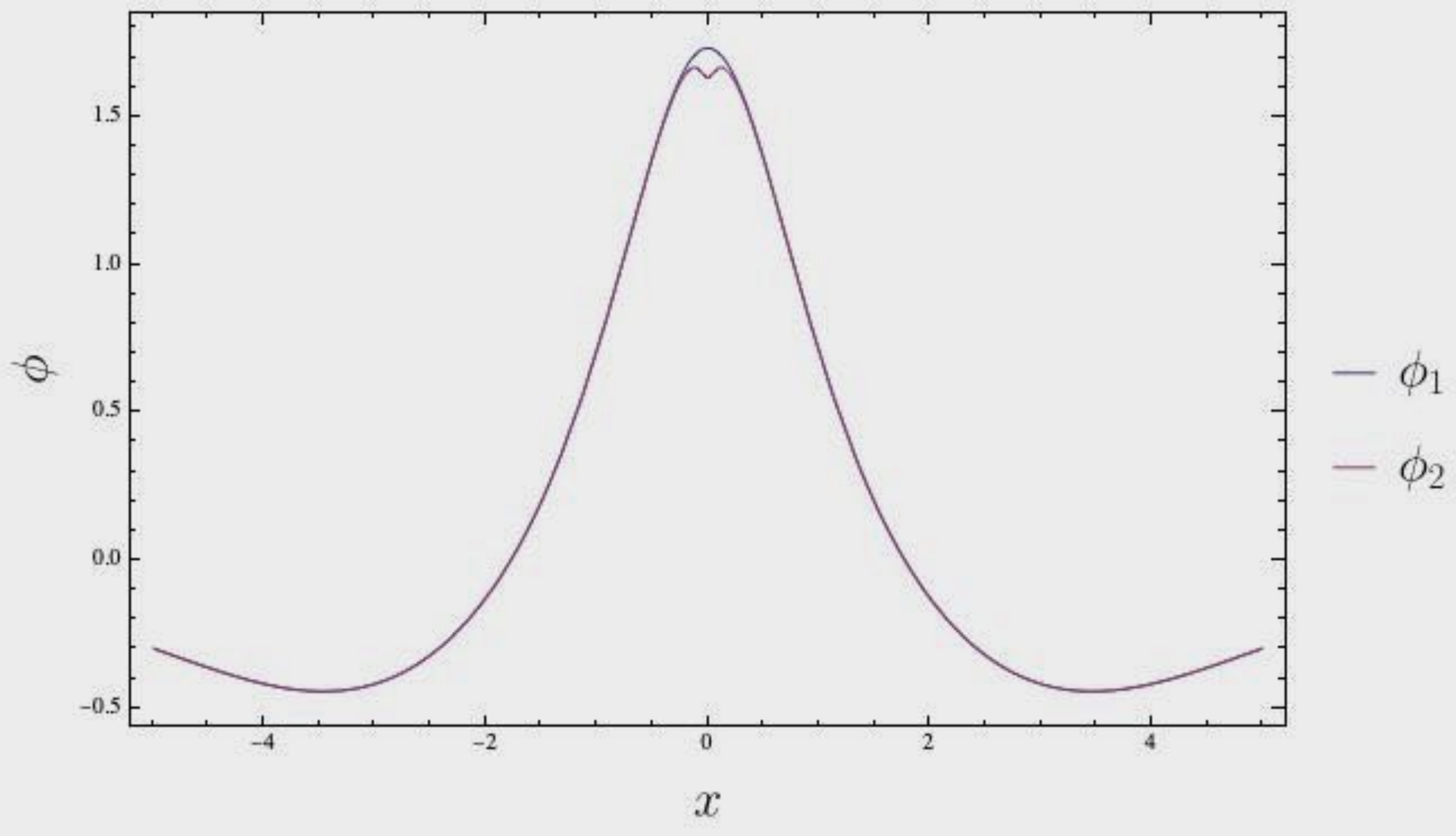
$$\phi_1(x,0) = \partial_x \arctan\left[\sqrt{3}\frac{\sin x}{\cosh\sqrt{3}x}\right]$$

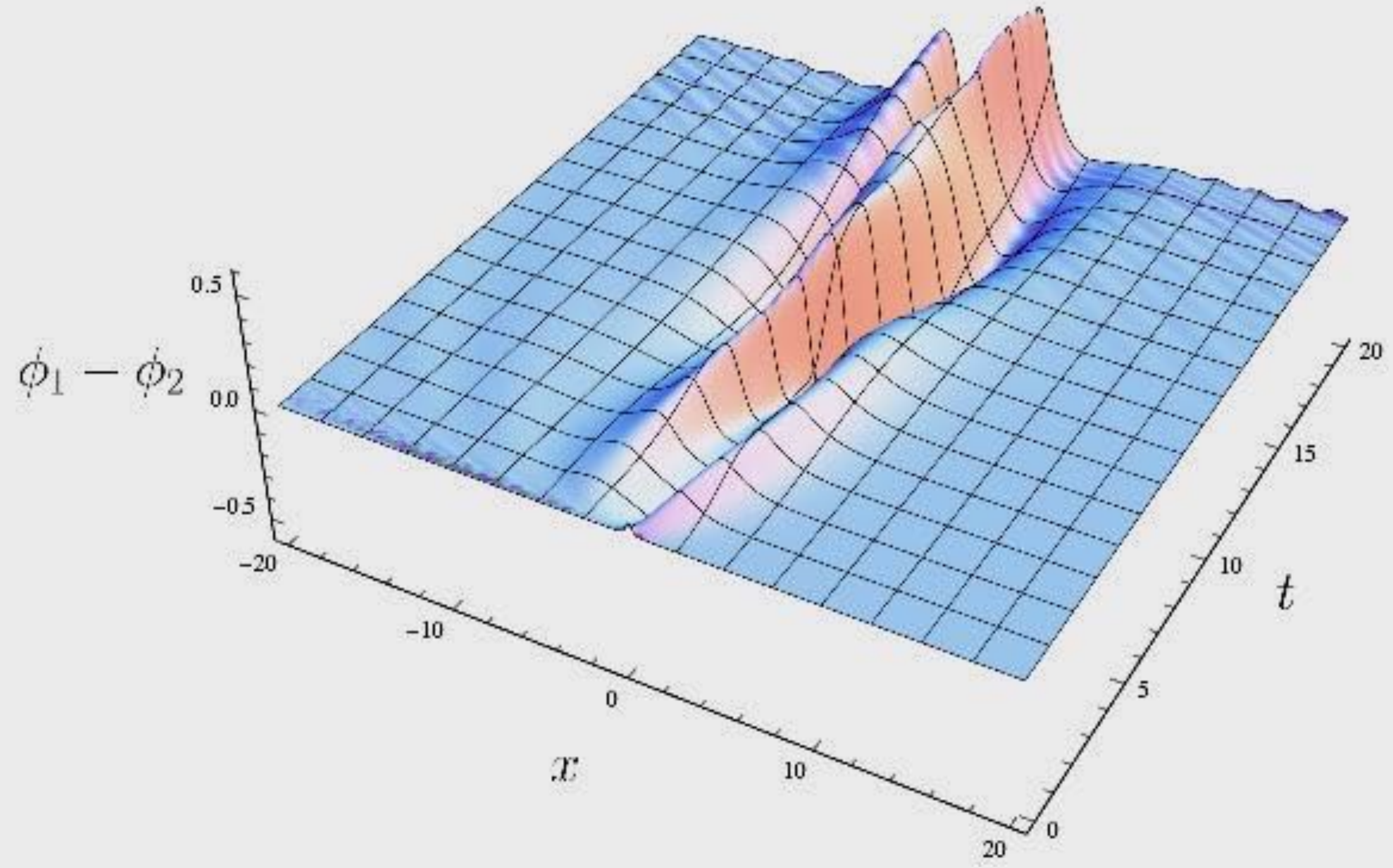
Evolves in t periodically: it is a breather solution

$$\phi_2(x,0) = \partial_x \arctan\left[\sqrt{3}\frac{\sin x}{\cosh\sqrt{3}x}\right] - \frac{1}{50}\frac{\cosh x \sin\frac{1}{10}}{\cosh 2x - \cos\frac{1}{5}}$$

Slightly deformed initial profile, with a small additional indentation around $x = 0$

INTEGRABLE FIELD THEORIES: THE MKDV EQUATION





Integrable Systems vs Deterministic Chaos

Integrable Field Theories: the mKdV equation